TRIGONOMETRY
KEEPING QUADRANTS STRAIGHT
LESS TO MEMORIZE!

by Dr. Stephen B. Soffer

Level: (11^{th} to 12^{h} grade or precalculus)
Tools needed: Scientific or graphing calculator

INTRODUCTION

Students of trigonometry often think that they have to memorize many different rules for dealing with trigonometric functions when the angles are outside of the familiar range between 0^o and 90^o. Here, we will simplify things and keep memorization to a minimum. We also deal with negative angles, larger angles, angles in degrees and radians, uses of calculators in trigonometry, and how graphs of trigonometric functions can help.

BASIC DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS

These basic definitions of the trigonometric functions were originally for right triangles, with other angles less than 90^o, but will lead to the ability to deal with angles in any of the four quadrants. Angles are written as the Greek theta (θ). The sides and the angle are shown in Figure 1.

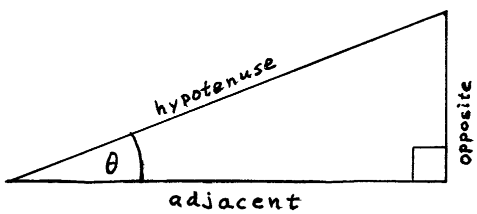

Fig. 1 Showing an angle and the three sides of a right triangle.

$$\sin\theta = \frac{opposite}{hypotenuse} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{opposite}{adjacent} \qquad \cot\theta = \frac{1}{\tan\theta}$$

These basic definitions must be remembered.

EXTENDING THE ANGLES TO 360°

Now for angles in the range $0 \leq \theta \leq 360^o$. Figure 2 shows angles in all four quadrants.

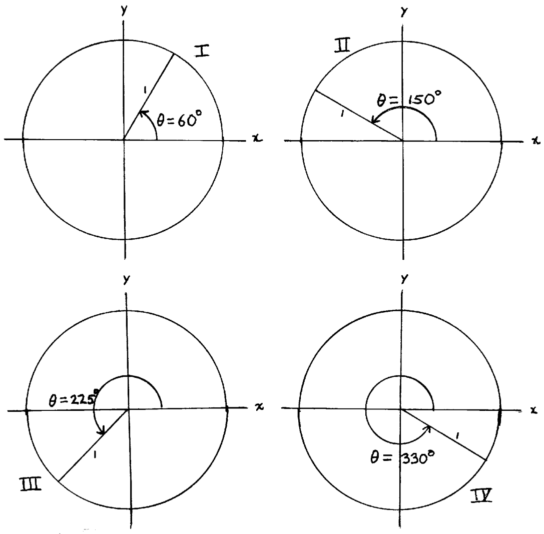

Fig. 2 Showing angles in all four quadrants

- A line of length one is used to show the direction of one side of any angle.
- Angles are measured *counterclockwise* from the *positive* *x*- axis.
- For an angle in any quadrant, look for the acute angle it makes with the positive or negative *x*-axis, whichever is nearest to the line of length one.
- To find the *sign* of a trigonometric function, think of:
  - "adjacent" as the *x* value of the line's end point
  - "opposite" as the *y* value of the line's end point
  - "hypotenuse" as +1.

  Then in the ratios given above for acute angles, the sign is positive if the numerator and denominator are of the same sign, and the sign is negative if the numerator and denominator are of opposite signs.

Examples: See Figure 2.

Quadrant I: $\theta = 60^o$ The angle is measured from the positive *x*-axis. It is already an acute angle. Both *x* and *y* of the line's endpoint are positive. So, all trigonometric functions are positive. If you are familiar with the standard triangle of angles 30, 60, and 90 degrees, the side lengths $1, \sqrt{3}, 2$ are opposite to those angles in the order given. The side 1 is adjacent, the side $\sqrt{3}$ is opposite, and 2 is the hypotenuse. From this:

$$\sin 60^o = \frac{\sqrt{3}}{2} \quad \cos 60^o = \frac{1}{2} \quad \tan 60^o = \frac{\sqrt{3}}{1} = \sqrt{3}$$

Quadrant II: $\theta = 150^o$ The negative *x*-axis is closest to the line. The acute angle with that axis is 180 - 150 = 30°. The *x* value of the end point of the line is negative (to the left); its *y* value is positive (upward). The standard triangle of 30, 60, and 90 degrees can be used again. This time the opposite is the side of length 1 and the adjacent is of length $\sqrt{3}$. A question mark (?) means we have to find the sign.

$$\sin 150 = ?\sin 30 = \frac{(+1)}{(+2)} = \frac{1}{2}$$

$$\cos 150 = ?\cos 30 = \frac{(-\sqrt{3})}{(+2)} = -\frac{\sqrt{3}}{2}$$

$$\tan 150 = ?\tan 30 = \frac{(+1)}{(-\sqrt{3})} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

Quadrant III: $\theta = 225^o$ The negative x-axis is closest to the line. The acute angle with that axis is 225 - 180 = 45°. The x value of the end point of the line is negative (to the left); its y value is negative (downward). The standard triangle of 45, 45, and 90 degrees can be used. The opposite and adjacent sides are of length 1. The hypotenuse is of length $\sqrt{2}$. A question mark (?) means we have to find the sign.

$$\sin 225 = ?\sin 45 = \frac{(-1)}{\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\cos 225 = ?\cos 45 = \frac{(-1)}{\sqrt{2}} = -\frac{1}{\sqrt{2}} = -\frac{\sqrt{2}}{2}$$

$$\tan 225 = ?\tan 45 = \frac{(-1)}{(-1)} = 1$$

Quadrant IV: $\theta = 330^o$ The positive x-axis is closest to the line. The acute angle with that axis is 360 - 330 = 30°. The x value of the end point of the line is positive (to the right); its y value is negative (downward). The same standard triangle as for 150° (the example in Quadrant II above) can be used. A question mark (?) means we have to find the sign.

$$\sin 330 = ?\sin 30 = \frac{(-1)}{2} = -\frac{1}{2}$$

$$\cos 330 = ?\cos 30 = \frac{\sqrt{3}}{2}$$

$$\tan 330 = ?\tan 30 = \frac{(-1)}{\sqrt{3}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

The examples so far used angles that were found in standard triangles. What if the angle is not one of those? You can use your scientific or graphing calculator. *If the angle is given in degrees, be sure that the calculator is in the degree mode!*

Example: $\theta = 160^o$: The signs are the same as in the example of 150° above. The calculator gives (after rounding to four decimal places: sin 160° = +sin 20° = 0.3420, cos 160° = -cos 20° = -0.9397, tan160° = -tan20° = -0.3640.

Student Activity 1

Find the sine, cosine, and tangent of these angles. If necessary, use your calculator in the degree mode and get values rounded to four decimal places. Even if you used a calculator, show that the values you get, including signs, agree with the rules given above.

(a) 30^o, (b) 45^o, (c) 300^o, (d) 240^o, (e) 280^o

If you have any trouble with this, see the section GRAPHING THE RULES.

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Discussion and Solutions

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DISCUSSION

Rather than treating each quadrant having its own set of rules, we start with minimal definitions and properties, and show that what happens in each quadrant follows naturally from them.

The question of reliance on calculators arises in current mathematics education. Before calculators, students used tables that were usually limited to the first quadrant (0 to 90 degrees) and had to figure out the extension to other quadrants. Modern scientific and graphing calculators allow entering angles of any size and sign as arguments of the sine, cosine, and tangent functions and the result "solves" the quadrant problem. Here we encourage also determining the results from basic principles.

The introduction of trigonometric functions as graphs allows another view. In fact, many of the applications of trigonometry go beyond surveying and finding the height of a flagpole, with sinusoids depending on time or position being important it their own right, with applications in electronics, wave motion, Fourier analysis, etc. Here we show the equivalence of the graphical view and the more traditional view using a rotating unit vector, tracing out the unit circle.

SUGGESTED SOLUTIONS OF STUDENT ACTIVITIES

We try to account for alternative approaches, but realize that your student may come up with other, equally valid ways of solving problems.

Student Activity 1

(a) This is a "standard" angle, and, since in the first quadrant, all signs are positive. The sides from smallest to largest are $1, \sqrt{3}, 2$ or any triangle proportional to that.

$$\sin 30^\circ = 1/2 = 0.5$$
$$\cos 30^\circ = \sqrt{3}/2 \approx 0.866$$
$$\tan 30^\circ = 1/\sqrt{3} = \sqrt{3}/3 \approx 0.577$$

(b) This is also a standard angle, with corresponding sides in the ratio $1:1:\sqrt{2}$. It is also in the first quadrant, so all signs are again positive.

$$\sin 45^\circ = \cos 45^\circ = 1/\sqrt{2} = \sqrt{2}/2 \approx 0.707$$
$$\tan 45^\circ = 1/1 = 1$$

(c) See Figure 7 in the student portion for a graphical construction.

The angle nearest the x-axis is 60 degrees and is measured to the positive x-axis. The signs are, symbolically:

sine: -/+ = -, cosine: +/+ = +, tangent: -/+ = -

$$\sin 300^\circ = -\sin 60^\circ = -\sqrt{3}/2 \approx -0.866$$
$$\cos 300^\circ = \cos 60^\circ = 1/2 = 0.5$$
$$\tan 300^\circ = -\tan 60^\circ = -\sqrt{3} \approx -1.732$$

(d) 240° lies in the third quadrant and makes an angle of 60 degrees with the negative *x*-axis. Signs are:

sine: -/+ = -, cosine: -/+ = -, tangent: -/- = +.

The absolute values are the same as in (c).

(e) 280° is also in the fourth quadrant and the angle of 80 degrees is not a standard angle. The signs are as in (c). Using a calculator with argument 80°:

$$\sin 280^o = -\sin 80^o \approx -0.985$$

$$\cos 280^o = \cos 80^o \approx 0.174$$

$$\tan 280^o = -\tan 80^o \approx -5.671$$