# LOGARITHMS – WHY BOTHER?
# SOME OF THEIR APPLICATIONS

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Level: Ninth grade to college
Purpose: To show that logarithms are more than just a topic students must learn to pass their mathematics courses and to introduce them to several applications.

## INTRODUCTION

**Logarithms** are taught as a topic in high school mathematics, but little attention is paid to their applications, especially with their diminished use as a computational aid. This lesson plan serves to fill that gap by covering several **applications** of logarithms. Some of this material is based on what is covered in my textbook, **Mathematics and Its Applications** (available under my original hard goods in TeachersPayTeachers.com.)

The student is assumed to be familiar with the basic properties of logarithms including: their relationship to exponents; logarithms of products, ratios and exponentials. (See also my mini-course LOGARITHMS – WHAT ARE THEY REALLY?). I start with some of the older uses, before modern calculators and computers took over most routine calculations. Then I cover several applications that have been around for a long time and are still in use.

Some problems are given, with their solutions separate, so you can exclude them in what you give the students.

## USING LOGARITHMS TO MULTIPLY OR DIVIDE NUMBERS WITH MANY DIGITS

Note to instructor: This section can be omitted if desired.

In the "old days" before accurate calculators and accessible, powerful computers, *slide rules* were used for multiplication, division, square roots, etc. Accuracy was often three or more significant figures. However, *tables* of logarithms were often used in a rather cumbersome process. I will give a brief introduction to these tables and a worked out application – multiplying two large numbers with a fairly large number of digits. The basic idea is that addition is quicker than multiplication where many digits are involved and logarithms allow use of addition. Similar comments apply to division and subtraction.

Tables of logarithms give what is called the **mantissa** – the *decimal* part of the logarithm. These tables are mostly for *common logarithms* – base 10. For example, a logarithm whose value is 3.2173 has a mantissa of .2173. The table I will use is from Thurman S.Peterson and Charles R. Hobby, **Intermediate Algebra For College Students, Sixth Edition** (Harper and Row, 1984).

To see how such tables are used for computations, consider the multiplication problem $26{,}529 \times 8{,}527$. While these are not very large numbers, standard multiplication methods requires five times four or 20 simple multiplications, followed by several additions – a process that is long and error prone. In scientific notation (more on that later), this is $2.6529 \times 10^4 \times 8.527 \times 10^3$. Tables of mantissas give only logs in the range [0,1), corresponding to original numbers in [1,10). That is why putting the numbers in scientific notation is useful for this application. The table only gives arguments to three figures, and mantissas to four. Table searching gives

$$2.65 \rightarrow .4232$$

$$8.53 \rightarrow .9309$$

(Interpolation would give more accurate results. I will skip this here.) Looking at the exponents of the original numbers the logarithms are 4.4232 and 3.9309, respectively.

Now, we can use the property that the log of a product of two numbers is the sum of the logs of the individual numbers. (Note: I will start using the abbreviation log part of the time.) Since adding is easier than multiplying, adding our two logarithms in the example gives 8.3541. Although tables of *antilogarithms* exist, using the same table as above and looking at the nearest mantissa, gives .3541 exactly – pure luck! Its corresponding value in the table is 2.26. Since the part of the logarithm before the decimal point – called the **characteristic** – is 8 , the result is multiplied by 10 to the eighth, giving approximately $2.26 \times 10^8$. We only got at best three digit accuracy. Now pulling out a modern graphing or scientific calculator, where * is the display symbol for multiplication, we get 26529*8527, yielding 22621783, which is approximately $2.262 \times 10^8$ . More dense tables and tables of antilogarithms give more accuracy and convenience. (The venerable reference, **Handbook of Mathematical Functions**, ... by M. Abromowitz and I. A. Stegun gives more accurate log tables, but not antilogs.)

I will not go into cases of numbers less than one, of division or exponentiation, as these topics now seem quaint and are mostly of historical interest.

## Slide Rules

These are basically **logarithmic scales**. The sliding parts are labeled in the *positions* of their logarithms. Multiplication is done by moving the "1" position of the slider to the first number over it on the fixed part, and looking over the position of the second number on the slider (with the help of a clear plastic slider containing a vertical line). The result is the number on the fixed part. You do not get the position of the decimal point or the power of 10 in scientific notation. This requires an order of magnitude estimate (more on that later). Division, exponentiation, square roots, cube roots, squares and cubes are also often supported. Some slide rules give transcendental functions such as trigonometric functions. A circular version gives more accuracy – up to five figures.

## Scientific Notation and Order of Magnitude

Now we get to topics that are still useful.

Scientific notation arises naturally from dealing with *very large* or *very small* numbers. It is also a way of dealing with *significant digits*, as we will see.

The standard form of scientific notation is

$$x.xxx\ldots \times 10^n$$

Here each *x* represents a *digit*. *n* is an integer, i.e.,

$$n \in \{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$$

In scientific *experiments*, only as many digits as can be accurately measured, or at least estimated, are included.

Example: A result of 6,638,912, where we really only trust the first four digits, can be written rounded in scientific notation as $6.639 \times 10^6$. Let's say this result is a length in meters. The same result in kilometers would be $6.639 \times 10^3$ kilometers or km. In centimeters, it would be written as $6.639 \times 10^8$ centimeters or cm. Notice how the same number precedes the power of 10, providing scientists a way to report their results in any metric system of units in a uniform way showing what they really believe to be accurate. (Unfortunately, this does not work in the English system or conversions between metric and English units, but most modern science uses a metric system.)

Example: 0.0000000375592, where only four digits are reliable would be written as $3.756 \times 10^{-8}$.

Here is a good place to discuss leading and lagging zeroes. In the second example, the seven zeroes in front have no bearing on significant digits. But, consider a result reported as $8{,}270.00 = 8.27000 \times 10^3$. The inclusion of the three lagging or following zeroes implies that we believe they are accurate. For this reason, the scientific notation is preferable as the usual form 8,270 is needed to show it is eight thousand and seventy, but if the last zero is not significant, it would be reported as $8.27 \times 10^3$.

**Order of magnitude** follows from the representation in scientific notation. A broad definition would be the *"nearest" power of 10* or, alternatively, the *exponent of 10* – i.e., the *common logarithm*. What constitutes "nearest" may be less clear. One possible way could be that if the value of $x.xxx\ldots$ before $10^n$ is less than 5, the order of magnitude is $10^n$ or just *n*. If it is 5 or greater, the order of magnitude would be $10^{n+1}$ or *n* + 1.

According to a Wikipedia article (resulting from a search in Alta Vista for "order of magnitude", one definition is to write the number in scientific notation and define the order of maginude as the exponent of 10. This is a *truncation* method.

Example: $65{,}721 = 6.5721 \times 10^4 \rightarrow$ order of magnitude is 4.

From the same article, *order of magnitude estimate* for a "variable whose precise value is unknown is an estimate rounded to the nearest power of ten". In the example given there, a number between 3 billion and 30 billion has an estimated value of 10 billion. They refer to rounding its logarithm with an order of magnitude between $10^{9.7}$ &$10^{10.6}$. According to my calculations, $\log_{10}(3 \times 10^9) \approx 9.477$ . I am not sure of the reasoning used.

Here is my proposal for an order of magnitude estimate. Consider the number in scientific notation of the form $x10^n$, where $1 \le x < 10$ , and $n$ is an integer. Then take its log to the base 10, giving

$$\log_{10}(x \cdot 10^n) = \log_{10} x + n\log_{10} 10 = \log_{10} x + n$$

We have $0 \le \log_{10} x < 1$. Denote $\log_{10} x \equiv l$ . Then, if $l < 0.5$ , the order of magnitude estimate is $n$. If $l \geq 0.5$ , the order of magnitude estimate is $n + 1$. Calculations give $10^{0.5} \approx 3162, \log_{10} 3162 \approx 0.49996$ . This means we can take 3.162 as a dividing line for the $x$ in scientific notation.

Examples: $314 \times 10^6 \rightarrow$ order of magnitude estimate of 6
$317 \times 10^6 \rightarrow$ order of magnitude estimate of 7

## SOME MORE APPLICATIONS OF LOGARITHMS

I will consider: logarithmic plots, the Richter scale for earthquake intensity, decibels in sound intensity (and other applications – see for example, from Alta Vista search on decibels – phys.unsw.edu.au/jw dB.html), and star magnitudes.

### Logarithmic Plots

I will start with an example from **Mathematics and Its Applications** – one of its applications, which gives more details than here.

The following table gives estimates of the way the Earth's population varied over time. It is followed by a figure showing one way to display the relationship between population and time. The sources are in the textbook.

| Year | Estimated Population |
|---|---|
| 10,000 B.C. | 3 million |
| 8,000 B.C. | 5.3 million |
| 3,000 B.C. | 100 million |
| 7 B.C. | 250 million |
| 1,650 A.D. | 500 million |
| 1,750 A.D. | 725 million |
| 1,850 A.D. | 1.175 billion |
| 1,900 A.D. | 1.600 billion |
| 1,950 A.D. | 2.564 billion |
| 1,980 A.D. | 4.478 billion |
| 1,991 A.D. | 5.423 billion |

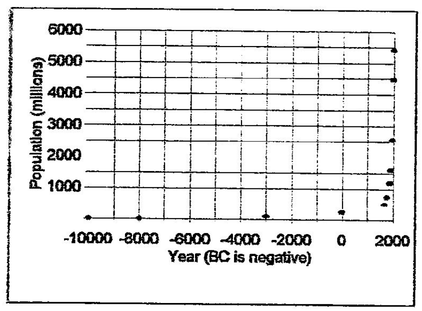


**Fig. 1.1** Linear plot of population vs. year

The figure is a simple linear plot of the data. It clearly shows very little change at first, followed by accelerated increase, which then becomes an explosive increase. These changes occur, respectively, roughly at 3,000 B.C. – "with the rise of some early urban civilizations", and about 1,750 A.D. – the start of the industrial revolution. The later points appear to fall almost on a vertical line.

"*Does this kind of display of data hide anything*?"