EXPONENTS AND POWERS
ELIMINATING THE CONFUSION

by Dr. Stephen B. Soffer

Level: (9^{th} to 12^{th} grade)
Tools needed: Scientific or graphing calculator
Related mini-course: *Logarithms*. Shows the relation to exponents.

INTRODUCTION

Students are often introduced to the properties of exponents and powers by means of a large number of rules. Here we keep the number of rules and definitions to a minimum and treat the properties as generally following in an obvious way from basic arithmetic in the case of exponents that are integers.

DEFINITIONS AND BASIC IDEAS

In arithmetic, we often multiply a number by itself. For example, to find the area of a square whose sides are each 4 centimeters long, we multiply 4 X 4 to get 16 square centimeters. We customarily write this as

$$4^2 = 16$$

Here the raised value or *superscript* 2 is the number of times 4 appears in the product. The number 4 is called the **base**. The value of the superscript 2 is the **exponent**. The result (16) is the **power**. In any *product* of two or more numbers, the individual numbers are the *factors* of the product. In this case the factors are equal in value and the factor is what we called the *base*.

Extending this idea to finding the volume of a cube whose sides are each 4 centimeters in length,

$$4 \times 4 \times 4 = 4^3 = 64$$

Here the base is still 4, the exponent is 3 and the power is 64. We sometimes also say this as: 4 "raised to the third power". Here we will mostly use the term exponent to avoid confusion.

A base with an exponent of 2 is called the *square* of that base. If the exponent is 3, we are said to have found the *cube* of that base. The relationship to the geometrical examples should be obvious.

In general,

$$\boxed{\text{base}^{\text{exponent}} = \text{power}}$$

Examples:

$3 \times 3 = 3^2 = 9$: base = 3, exponent = 2, power = 9

$10 \times 10 \times 10 = 10^3 = 1{,}000$: base = 10, exponent = 3, power = 1,000

Student Activity 1

For each given number in the form given, identify the base and exponent and evaluate the power.

(a) 16^2, (b) 5^4, (c) 8^3

USE OF THE CALCULATOR

Most scientific and graphing calculators use either a y^x key or a carat (^) key to find powers. We will show keystrokes separated by commas. The commas are not entered. Results are usually gotten by a "=" or ENTER key.

Example: To evaluate 3^4, enter: 3, y^x or ^, 4,= or ENTER. The result is to display 81.

Student Activity 2

Use a scientific or graphing calculator to evaluate the following:

(a) 15^4, (b) 20^3, (c) 17^5, (d) 1^5, (e) 5^1, (f) 2^{10}, (g) 2^{40}

In this activity, what happened with the last one (g)? You may have seen something like: $1.099511628\ ^{12}$ or 1.099511628E12. These are both ways of saying the value is $1.099511628 \times 10^{12}$. Here the calculator usually gives an *approximate* value. When the result of a calculation is very large or very small the result is usually given in what is called **scientific notation**. We will see more about that later.

PRODUCTS OF POWERS WITH EQUAL BASES

Consider the product $4^2 \times 4^3$, which we may write as $4^2 \cdot 4^3$ or simply $4^2 4^3$. Write out each power as a product of its individual bases. In other words we want to write it the long way so you will understand the principle.

$$4^2 \cdot 4^3 = (4 \cdot 4) \cdot (4 \cdot 4 \cdot 4) = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$$

We removed the parentheses, as these were used just to show the original groups of factors. By simple counting, we see that 4^2 contributes two factors of 4 and 4^3 contributes three more. Together they give $2 + 3 = 5$ factors of 4. Summarizing in exponential notation:

$$4^2 \cdot 4^3 = 4^5$$

As another example, you can show in the same way that $3^4 \cdot 3^3 = 3^7$

From these examples, we can easily see that, for any base x, and any two exponents that are *positive integers*, the products of the powers x^m and x^n is

$$\boxed{x^m \cdot x^n = x^m x^n = x^{m+n}}$$

We have used notation with and without the dot to show multiplication. Note that x does not have to be an integer. We will see later that m and n can be values other than positive integers. But it should be clear how this rule works for exponents that are positive integers.

Student Activity 3

For each product of powers, get it expressed as a single power with the same base. Evaluate the single power, using a calculator if needed. Check by evaluating the individual powers that were factors of the overall product and multiplying them. Which was easier?

(a) $2^3 \cdot 2^4$, (b) $3^4 \cdot 3^2$, (c) $10^2 \cdot 10^3$, (d) $2^2 \cdot 2^3 \cdot 2^4$ (Hint: Apply the rule twice.), (e) $2.5^2 \cdot 2.5^3$

The last item points out that the base does not have to be a positive integer.

EXPONENTS AND POWERS
ELIMINATING THE CONFUSION

Discussion and Solutions

by Dr. Stephen B. Soffer

DISCUSSION

Students should be encouraged to derive the rules – at least for exponents that are integers. (See also the discussion in CLOSING COMMENTS.) Calculators can be used to get numerical results, especially for non-integer bases or exponents, but you can encourage them to use "mental" arithmetic where possible. Calculators can also be used to verify that the rules give the same results (and more simply) than calculating the individual powers and then multiplying or dividing (perhaps requiring storage in memory). In some of the solutions here, we give alternative approaches, often using a calculator, and give the key strokes. We will generally omit the commas between strokes, we use the carat for powers, and it is understood that all calculator entries end with = or ENTER. Understanding exponents is essential for *Logarithms* – a separate lesson plan.

SUGGESTED SOLUTIONS OF STUDENT ACTIVITIES\

Student Activity 1

(a) The base is 16. The exponent is 2. The power is 256.

(b) The base is 5. The exponent is 4. The power is 625.

(c) The base is 8. The exponent is 3. The power is 512.

Student Activity 2

(a) 50,625 (key strokes: 15^4). (b) 8,000. (c) 1,419,857. (d) 1. Unity to any non-zero power returns unity. (e) 5. Any base with unity exponent evaluates to the base. (f) 1,024. (g) 1.099511628E12 (on a TI-83).

Student Activity 3

(a) $2^3 \cdot 2^4 = 2^{3+4} = 2^7 = 128$ Key strokes: 2^7
Doing it the long way: $2^3 = 8$, $2^4 = 16$. Their product is 128.

(b) $3^4 \cdot 3^2 = 3^{4+2} = 3^6 = 729$. With individual powers, $81 \times 9 = 729$

(c) $10^2 \cdot 10^3 = 10^5 = 100{,}000$. Or $100 \times 10000 = 100{,}000$

(d) $2^2 \cdot 2^3 \cdot 2^4 = 2^2 \cdot 2^{3+4} = 2^2 \cdot 2^7 = 2^{2+7} = 2^9 = 512$. By the associative property for multiplication, we could also combine the first two powers in the first step, giving 2^5. With individual powers, $4 \times 8 \times 16 = 512$.

(e) $2.5^2 \cdot 2.5^3 = 2.5^{2+3} = 2.5^5 = 97.65625$. Individually, 6.25 times 15.625 gives the same result.