CHAOS – AN INTRODUCTION
LEARNING BY DOING

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Level: From ninth grade up – for any students who are familiar with use of a scientific or graphing calculator.

Purpose: To introduce students to this fascinating contemporary subject by exploring it directly.

Companion Lesson Plan: On applying chaos to encryption.

INTRODUCTION

The idea of **chaos** has been around for a number of years since it emerged as a popular subject. The purposes of this lesson plan are:

- To develop an accessible explanation and definition of what is meant by *chaos*.
- To get students to experience it first hand by explorations or simulations on a scientific or graphing calculator.
- To learn to analyze results of such simulations.
- Encourage students to suggest and follow up on new directions for research.
- Suggest physical manifestations of chaotic phenomena that can be tried in experiments, such as a dripping faucet.

This lesson plan is not primarily intended for a high level course or to cover every possible detail. To extend it further in depth and extent, the list of references to books and Internet sources can be consulted for further details. Here, we hit the highlights and stress exploration and experimentation more than pure theory. This makes it adaptable to a variety of levels. It can be part of a self-contained course in high school, such as for self study or a group project (before calculus) or part of a more advanced calculus level course in chaotic dynamics for later high school grades, AP courses or college courses.

A scientific calculator can be used. However, having basic knowledge of and access to a graphing calculator helps for efficiency and depth of the explorations.

WHAT IS THIS THING CALLED "CHAOS"?

Note: Instructors can omit some of the more formal aspects.

For a start, let's say that there is no single universal definition of **chaos**. To start to see what it is, we will start with the idea of *iteration*.

Here is what is generally meant by iteration:

1. Start with a real (or sometimes complex) number – often called the *seed*. Give it the name x_0.

2. Have some *rule* that gives another value after x_0. Call the new value x_1. In general, this can be denoted by
   $$x_1 = f(x_0)$$
   where $f(x)$ is a function of *x*.

3. Apply the rule over and over again to generate more numbers.
   $$x_2 = f(x_1)$$
   $$x_3 = f(x_2)$$
   $$x_4 = f(x_3)$$
   $$etc.$$

The resulting *sequence* of numbers can be written as

$$x_0, f(x_0), f^2(x_0), f^3(x_0), f^4(x_0), \ldots$$

where the superscripts after *f* are *not* exponents, but rather represent *composite functions*. For example, using two common notations

$$f^3(x_0) = f(f(f(x_0))) = f \circ f \circ f(x_0)$$

[Some sources may prefer to enclose the superscript in parentheses, such as $f^{(3)}(x_0)$ to avoid confusion with exponents.]

Although we used the notation x_i – implying real values, the methods extend to the complex plane with x_i replaced by $z_j = x_j + iy_j$ – using the iteration index *j* to avoid confusion with the imaginary number unit *i*. [Alternatively, the index *n* is often used.]

Example: Exponential Population Growth and Decay

Consider a well known model for population growth and decay. Let P_i denote the population of a particular species (say, within a given ecosystem) in a year denoted by i. Initially the population is P_0. The basic assumption is that the population each year is a *constant multiple* of the population in the preceding year. Calling this multiple *m*, we get the rule

$$P_{i+1} = mP_i, i = 0, 1, 2, \ldots$$

[Some prefer $P_i = mP_{i-1}, i = 1, 2, 3, \ldots$]

Applying the rule to an initial population P_0

$$P_1 = mP_0$$

$$P_2 = mP_1 = m(mP_0) = m^2 P_0$$

$$P_3 = mP_2 = m(m^2 P_0) = m^3 P_0$$

It is easy to see that, in general

$$P_i = m^i P_0, i = 0,1,2,3,\ldots$$

Note: The exponential or geometric growth model also has applications in finance, with the theory of compound interest of a single investment left to grow.

The last way of representing the process is called a *nonrecursive* sequence. We do not have to use the rule iteration by iteration, but can jump right to the i^{th} value.

The kind of behavior depends on the value of the parameter *m*.

If $m > 1$, the population grows exponentially, getting larger at an increasing rate without any upper limit.

If $m < 1$ (but positive), the population *decays* exponentially, getting smaller and tending towards zero. [At some point, the model will start to indicate a population less than one. Obviously, there is "graininess" and the continuous model must be rounded to an integer value. For example, if the population is calculated to be less than 0.5, it can be regarded as extinct.]

If $m = 1$, the population stays constant.

Example: The Logistic Parabola

Consider the rule

$$x_{i+1} = rx_i(1 - x_i), i = 0,1,2,3,\ldots$$

r is a constant – the single parameter of the model.

This is like the exponential growth model, but with the extra factor $(1 - x_i)$. Using the distributive principle,

$$x_{i+1} = rx_i - rx_i^2$$

This shows that the extra factor produces a *quadratic* model. More generally, it has led to a **nonlinear** model. [We will see that the nonlinearity is responsible for much of the complexity of these sort of processes.] Superficially, the extra factor makes the model look only slightly more complex. Actually, it can cause an *enormous* increase in the complexity of the behavior it produces, as we will see .

Preliminary Exploration of the Logistic Model

We will look at $r > 0$. Also, to make it more directly applicable to a population, we will take the viewpoint that we are looking at a closed and finite ecosystem, where there are not enough resources to support a very large population. Imagine, say, and island or a valley surrounded by steep mountains on all sides. If we let P_{max} be the maximum possible population and P_i be the population at any iteration, it is natural to define an iteration as a generation (or a year as before) and we can let x_i be the ratio of the population to its maximum possible in any given iteration (or generation)

$$x_i = \frac{P_i}{P_{max}} \rightarrow P_{i+1} = rP_i(1 - P_i / P_{max})$$

From this, it follows that x_i is limited to

$$0 \leq x_i \leq 1$$

We can make a stronger statement of the range by noting that at either 0 or 1, the population will die out in the next iteration. At 0, there is no member left to produce the next generation. At 1, the scramble for limited resources has led to warfare, cannibalism, etc. and the population will die out also. Our stronger statement is

$$0 < x_i < 1$$

The table gives a few results. (Values giving the fifth decimal place may be rounded.) Details of how these tables were produced are given in Appendix A.

r	x_0	i	x_i
2	0.50	0	0.5
		1	0.5
		2	0.5
		3	0.5
2	0.25	0	0.25
		1	0.375
		2	0.46875
		3	0.49805
		4	0.49999
		5	0.50000
		6	0.50000
		7	0.50000

The table is continued.

r	x_0	i	x_i
2	0.75	0	0.75
		1	0.375
		2	0.46875
		3	0.49805
		4	0.49999
		5	0.50000
		6	0.50000

What have we learned so far? So far, it seems like, regardless of the starting value, we end up at 0.5 (or very close to it). The "path" represented by the sequence $x_0, x_1, x_2, ...$ is called the **orbit** of the sequence. The value of 0.5 is an **attractor**. Other terms are *strange attractor* and *stable fixed point*. There may be different interpretations of these terms, but basically, it means that the orbit settles down to at or near a single value after a sufficient number of iterations. Paths starting with 0.25, 0.5 or 0.75 tend to it as an "equilibrium" value – half the maximum allowed ratio.

Next, consider an orbit starting at 0.50, but with a parameter value of $r = 3.0$. The next table shows its behavior.

r	x_0	i	x_i
3	0.50	0	0.50
		1	0.75
		2	0.5625
		3	0.73828
		4	0.57967
		5	0.73096
		6	0.58997
		7	0.72571
		8	0.59716

Although only eight iterations are shown, it appears that the orbit is starting to oscillate between values of approximately 0.6 and 0.7. More iterations would verify if this observation is true and what the values are.

The next table skips ahead and shows some of the higher value iterations.

r	x_0	i	x_i
3	0.50	50	0.63425
		51	0.69593
		52	0.63483
		53	0.69546
		54	0.63538
		...	...
		100	0.64330
		101	0.68839
		102	0.64352
		103	0.68820
		104	0.64374
		...	...
		150	0.64747
		151	0.68476
		152	0.64759
		153	0.68465
		154	0.64771

It looks like the orbit is getting close to oscillating between about 0.65 and 0.68. However, the values seem to be getting closer together. We look at some still higher iterations.

r	x_0	i	x_i
3	0.50	300	0.65302
		301	0.67976
		302	0.65306
		303	0.67972
		304	0.65311

Although it appears to be settling down to about 0.65 and 0.68, you would really need to go further to be sure they are not approaching each other. As far as we have gone, the behavior suggests a *periodic* orbit, with a period of one (iteration).

Now, increasing the model parameter still further to $r = 3.5$, we see the orbit shown next.

r	x_0	i	x_i
3.5	0.50	0	0.50
		1	0.875
		2	0.38281
		3	0.82693
		4	0.50090
		5	0.87500
		6	0.38282
		7	0.82694
		8	0.50088
		9	0.87500
		10	0.38282
		11	0.82694
		12	0.50088

Compare the results for iterations 0,4,8,12. Similarly, compare 2,6,10 and 3,7,12. In these groups the values are quite similar. This suggests an orbit of period 4 might be developing. It appears that there is a cycle of about 0.5, 0.875, 0.383, 0.827. Checking at a higher number of iterations, starting with the 100^{th}, we get 0.50088,0.87500,0.38282,0.82694 to five places at least.

For $r = 3.5$, the behavior was a bit complicated, but certainly not random. Anyone seeing the first 12 iterations could confidentially predict what happens next to five figure accuracy at least. Now we move to $r = 3.99$. We will see in a while that there is a practical upper limit of 4 for the model parameter. So, this value is close to that limit. The next table shows some of its earlier iterations for a starting value of 0.25.

r	x_0	i	x_i
3.99	0.25	0	0.25
		1	0.74813
		2	0.75185
		3	0.74442
		4	0.75914
		5	0.72956
		6	0.78723
		7	0.66831
		8	0.88446
		9	0.40773
		10	0.96353
		11	0.14022
		12	0.48103

The table continues.

r	x_0	i	x_i
3.99	0.25	13	0.99606
		14	0.01564
		15	0.06144
		16	0.23007
		17	0.70678
		18	0.82689
		19	0.57113
		20	0.97731

What should we make of this behavior? At first, it appears that there might be an attractor at x = 0.75. However, after a few iterations (mainly about the first 10), it looks like the orbit has an oscillation with increasing amplitude. After this, the behavior seems quite *random*, with no apparent periodic behavior, let alone being attracted to a given fixed point. This is called **chaos**.

Now, let's see what happens if we change the starting value x_0 by a small amount. In particular, consider a starting value of 0.2501. We show the values side by side in the next table.

r	x_0	i	x_i	x_0	i	x_i
3.99	0.25	0	0.25	0.2501	0	0.2501
		1	0.74813		1	0.74832
		2	0.75185		2	0.75146
		3	0.74442		3	0.74521
		4	0.75914		4	0.75759
		5	0.72956		5	0.73276
		6	0.78723		6	0.78134
		7	0.66831		7	0.68169
		8	0.88446		8	0.86579
		9	0.40773		9	0.46363
		10	0.96353		10	0.99222
		11	0.14022		11	0.03079
		12	0.48103		12	0.11906
		13	0.99606		13	0.41849
		14	0.01564		14	0.97099
		15	0.06144		15	0.11238

Here the starting values differ by 0.0001 or a 0.04% difference based on 0.25. We see that for the first few iterations we have similar behavior of both orbits, generally increasingly large swings, but with comparable values. As we go to the region where chaos starts (about the 12th iteration), we see not only randomness for each individual orbit, but the two orbits lose all resemblance to each other.

This makes you wonder what would happen if the starting values were even closer together. (At this point, you may want to get your students to start thinking ahead.) To check this, consider inserting six zeroes between 0.25 and the final decimal digit of 1. This makes for a difference of 0.000000001 or $4 \times 10^{-7}\%$ from 0.25. The results are shown in the table with selected iterations. After the starting value, results are rounded to five places.

r	x_0	i	x_i	x_0	i	x_i
3.99	0.25	0	0.25	0.250000001	0	0.250000001
		1	0.74813		1	0.74813
		2	0.75185		2	0.75185
		3	0.74442		3	0.74442
		4	0.75914		4	0.75914
		5	0.72956		5	0.72956
		6	0.78723		6	0.78723
		7	0.66831		7	0.66831
		8	0.88446		8	0.88446
		9	0.40773		9	0.40773
		10	0.96353		10	0.96353
		11	0.14022		11	0.14022
		12	0.48102		12	0.48103
		13	0.99606		13	0.99606
		14	0.01564		14	0.01564
		15	0.06144		15	0.06145
		16	0.23007		16	0.23011
		17	0.70678		17	0.70686
		18	0.82689		18	0.82677
		...	...		...	...
		25	0.98715		25	0.98283
		26	0.05059		26	0.06734
		27	0.19166		27	0.25060
		28	0.61815		28	0.74932
		29	0.94180		29	0.74948
		30	0.21871		30	0.74916
		31	0.68179		31	0.74979

The table is continued.